

Revivals of Zitterbewegung of a bound localized Dirac particle

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We show that a bound localized Dirac particle exhibits a revival of the Zitterbewegung (ZB) oscillation amplitude. These revivals go beyond the known quasiclassical regenerations in which the ZB oscillation amplitude is decreasing from period to period. We study this phenomenon in a Dirac-oscillator and show that it is possible to set up wave-packets in which there is a regeneration of the initial ZB amplitude.

I. INTRODUCTION

In the context of relativistic quantum mechanics there is a surprising phenomenon introduced by Schrödinger in 1930 as Zitterbewegung (ZB) [1]. He showed that there is a rapid “trembling” motion of a Dirac particle around its otherwise rectilinear average trajectory, due to the interference between negative and positive energy eigenvalues. There has been a lot of theoretical studies of ZB, but not a direct observation, due to the fact that the predicted frequency and amplitude are impossible to measure experimentally at present. Lock showed that ZB has a transient character for a free localized Dirac particle pointing out that ZB effect for a localized wave-packet in an external field depends on the eigenvalues of the Hamiltonian [2]. Nowadays, there is an intense interest in ZB of electrons in semiconductors (see the review of Zawadzki and Rusin [3] and references therein). Recently, ZB has been studied in graphene [3–11] where it has been related to electric conductivity. In particular, revivals and ZB were studied in the electric current in monolayer graphene in a perpendicular magnetic field [10]. In 2010, Gerritsma et al. [12] simulated experimentally the electron ZB by means of trapped ions and laser excitations adjusting experimentally some parameters of the Dirac equation.

On the other hand, quantum revival of wave-packets is an interference quantum phenomenon related to the temporal evolution of wave-packets, relativistic and nonrelativistic. Quantum revivals have been investigated theoretically including in atomic, molecular and nonlinear systems [13–19] and observed experimentally in a lot of different quantum systems, such as Rydberg atoms and molecules, and Bose-Einstein condensates [16, 20].

In what follow we show that there is a revival of ZB oscillations amplitude when a bound Dirac-electron is considered. We have chosen a Dirac oscillator to analyze this behavior because it is exactly soluble and it is a model that has applications in several branches of physics (see [21] and references therein). In this work it is demonstrated that besides ZB and quasiclassical oscillations studied previously by other authors [22], there exists a revival or regeneration of the ZB oscillations amplitude.

To describe quantum revivals, let us consider an initial wave-packet that is a superposition of eigenstates localized around some energy level E_{n_0} , it is appropriate to expand the energy around n_0 if $|n - n_0|/n_0 \ll 1$,

$$E_n \approx E_{n_0} + E'_{n_0}(n - n_0) + \frac{E''_{n_0}}{2}(n - n_0)^2 + \dots \quad (1)$$

and each term in the series defines an important time scale, $T_{CL} = \frac{2\pi\hbar}{|E'(n_0)|}$, $T_R = \frac{2\pi\hbar}{|E''(n_0)|/2}$ where T_{CL} is associated with the classical periodic motion of the wave-packet and T_R is the revival time (the validity of this expansion has been demonstrated in [14, 24, 25]). The wave-packet initially evolves quasiclassically with period T_{CL} , then spreads and collapses, but at later times, around T_R the wave-packet regenerates and reaches approximately its initial shape. For times that are rational fractions of T_R wave-packets split in clones of themselves [16, 23]. After the revival time a new cycle starts with quasiclassically behavior, collapses, fractional revivals and revivals. Revivals are usually analyzed using the autocorrelation function $A(t)$, which is the overlap between the initial and the time-evolving wave-packet. An alternative approach in terms on uncertainty entropic relations was proposed [26].

II. REVIVALS OF ZITTERBEWEGUNG IN A DIRAC OSCILLATOR

An appropriate system to discuss revivals of ZB for bounded states is a (2+1)-Dirac oscillator, due to the fact that it is exactly soluble and allows us to study this phenomenon in a simple system. So, we shall consider the Hamiltonian for a Dirac oscillator [27] with frequency ω

$$H = c\alpha \cdot (\mathbf{p} - im\omega\beta\mathbf{r}) + \beta mc^2 \quad (2)$$

where m is the rest mass of the Dirac particle (p. e. an electron), α and β are the Dirac matrices, and c the speed of light. We shall introduce the complex coordinate as in [21] $z = x + iy$ and using the usual creation and annihilation operators notation in terms of z and \bar{z}

$$a = \frac{1}{\sqrt{m\omega\hbar}}p_{\bar{z}} - \frac{i}{2}\sqrt{\frac{m\omega}{\hbar}}z$$

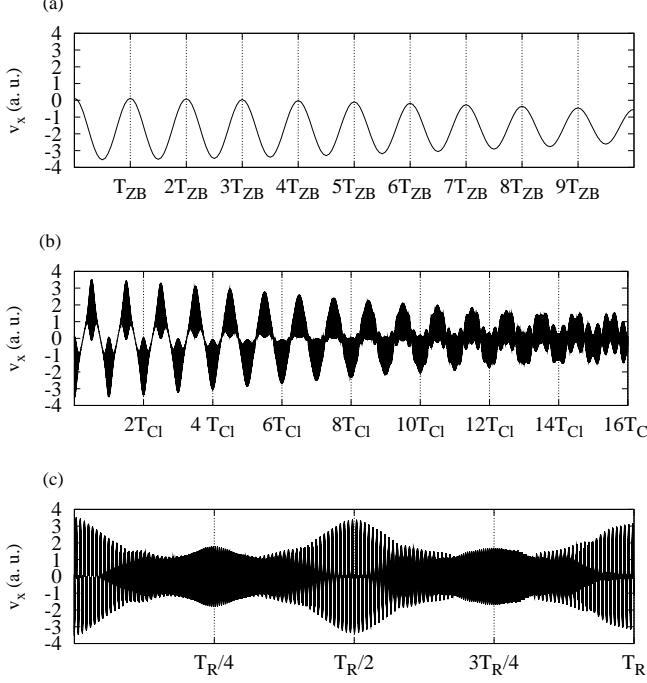


FIG. 1. Time dependence of $\langle v_x \rangle$ for the initial wave-packet with $\sigma = 3$, $n_0 = 30$, and oscillator frequency $\omega = 10^3$, for which $T_{ZB} = 6.15 \times 10^{-5}$, $T_{CL} = 8.54 \times 10^{-3}$, and $T_R = 1.19$ (all in a. u.). The vertical dotted lines stand for (panel a) T_{ZB} periods (panel b) T_{CL} periods, and (panel c) T_R periods.

$$a^+ = \frac{1}{\sqrt{m\omega\hbar}}p_z + \frac{i}{2}\sqrt{\frac{m\omega}{\hbar}}\bar{z}$$

the Hamiltonian reads

$$H = \begin{pmatrix} mc^2 & 2c\sqrt{m\omega\hbar}a^+ \\ 2c\sqrt{m\omega\hbar}a & -mc^2 \end{pmatrix}. \quad (3)$$

It is not difficult to show that the energy eigenfunctions are given by

$$|\phi_n^\pm\rangle = \begin{pmatrix} \pm\sqrt{\frac{1}{2} \pm \xi_n}|n\rangle \\ \mp\sqrt{\frac{1}{2} \mp \xi_n}|n-1\rangle \end{pmatrix}, \quad (4)$$

with

$$\xi_n = \frac{1}{2\sqrt{1 + \frac{4\hbar\omega n}{mc^2}}} \quad (5)$$

and with $n = 0, 1, \dots$ and the energy spectrum is, in turn,

$$E_n^\pm = \pm mc^2 \sqrt{1 + \frac{4\hbar\omega n}{mc^2}}. \quad (6)$$

We shall construct a superposition state of two wave-packets as the initial particle wave-packet

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\Psi_-\rangle + |\Psi_+\rangle) \quad (7)$$

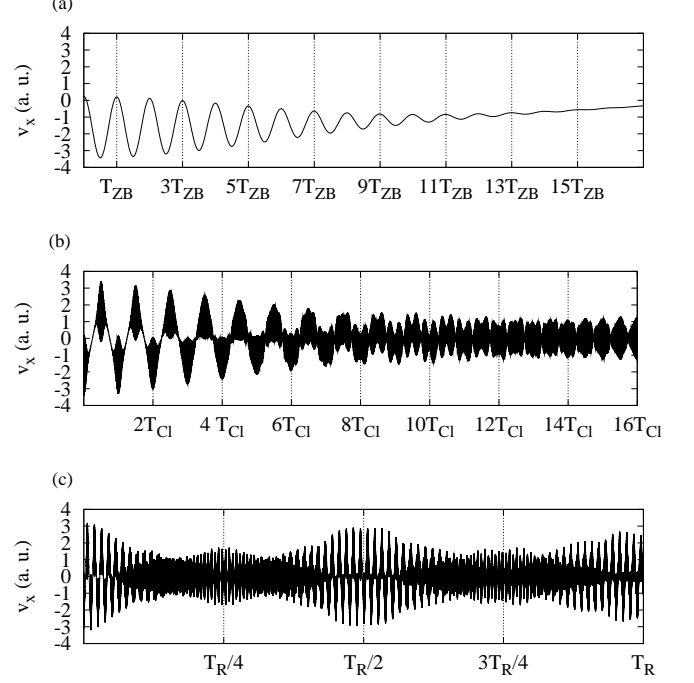


FIG. 2. Time dependence of $\langle v_x \rangle$ for the initial wave-packet with $\sigma = 3$, $n_0 = 15$, and oscillator frequency $\omega = 10^3$, for which $T_{ZB} = 8.16 \times 10^{-5}$, $T_{CL} = 6.4 \times 10^{-3}$, and $T_R = 0.5$ (all in a. u.). The vertical dotted lines stand for (panel a) T_{ZB} periods (panel b) T_{CL} periods, and (panel c) T_R periods.

where the above wave packets are defined as the linear combination

$$|\Psi_+\rangle = \sum_n c_n^+ |\phi_n^+\rangle \quad \text{and} \quad |\Psi_-\rangle = \sum_n c_n^- |\phi_n^-\rangle, \quad (8)$$

each of them centered around a given eigenvalue $E_{n_0}^+$ and $E_{n_0}^-$, respectively, with coefficients Gaussianly distributed ($c_n^+ = c_n^- = c_n$) as

$$c_n = \sqrt{\frac{1}{\pi\sqrt{\sigma}}} e^{-(n-n_0)^2/2\sigma}. \quad (9)$$

We can write the temporal evolution of the initial wave-packet as

$$|\Psi_0(t)\rangle = \frac{1}{\sqrt{2}} \sum_n (c_n^+ |\phi_n^+\rangle e^{iE_n^+ t/\hbar} + c_n^- |\phi_n^-\rangle e^{iE_n^- t/\hbar}) \quad (10)$$

taking into account that

$$|\Psi_\pm(t)\rangle = \sum_n (c_n^\pm |\phi_n^\pm\rangle e^{iE_n^\pm t/\hbar}). \quad (11)$$

The series expansion (1) should be interpreted in the context of the temporal evolution of the wavepacket. If we replace the value E_n by the expansion (1) in Eq. (11)

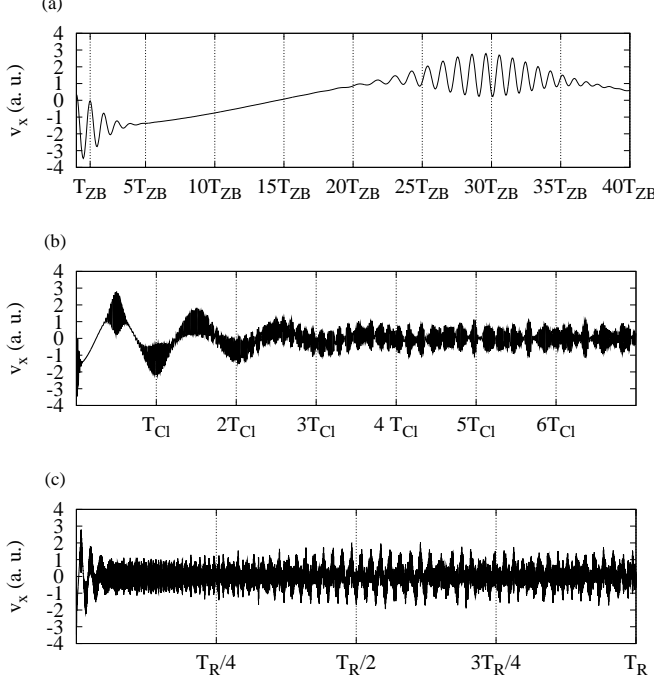


FIG. 3. Time dependence of $\langle v_x \rangle$ for the initial wave-packet with $\sigma = 20$, $n_0 = 10$, and oscillator frequency $\omega = 10^3$, for which $T_{ZB} = 9.46 \times 10^{-5}$, $T_{CL} = 5.55 \times 10^{-3}$, and $T_R = 0.33$ (all in a. u.). The vertical dotted lines stand for (panel a) T_{ZB} periods (panel b) T_{CL} periods, and (panel c) T_R periods.

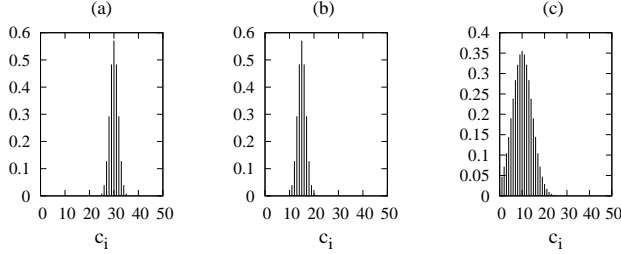


FIG. 4. Coefficients (Eq. 9) for (a) $n_0 = 30$, $\sigma = 3$, (b) $n_0 = 15$, $\sigma = 3$, and (c) $n_0 = 10$, $\sigma = 20$.

we can see that each term in the exponential (except the first) defines an important characteristic time scale. The first term is unimportant because it is an overall phase. The following two terms define the classical periodicity, and the revival time [14, 24, 25].

Therefore the corresponding classical period and the revival time for $|\Psi\rangle$ yield straightforwardly

$$T_{CL} = \frac{\pi}{\omega} \sqrt{1 + \frac{4\hbar\omega}{mc^2} n_0} \quad (12)$$

and

$$T_R = \frac{\pi mc^2}{\hbar\omega^2} \left(1 + \frac{4\hbar\omega}{mc^2} n_0\right)^{3/2}. \quad (13)$$

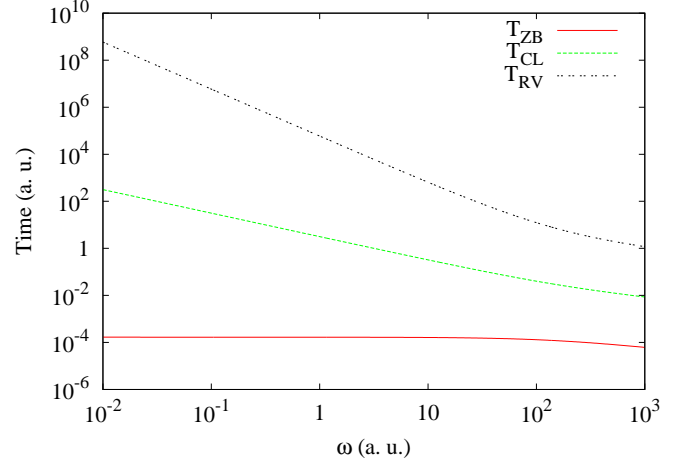


FIG. 5. (Color online) Temporal scales T_{ZB} , T_{CL} and T_R vs ω (all in a. u.) for $n_0 = 30$.

To calculate the period of ZB we shall determine the temporal evolution of the x and y components of the velocity which are given by $\langle v_j \rangle = \langle i[H, r_j]/\hbar \rangle$, ($j = x, y$), where σ_x and σ_y are the Pauli matrices. For the wave-packet $|\Psi_0\rangle$, after some algebra and taking into account that $|n\rangle$ is an orthonormal set, the temporal evolution for the velocities is given by:

$$\begin{aligned} \langle v_x \rangle &= 2 \sum_{n=0}^{\infty} c_n c_{n+1} (\eta_n \cos((E_n + E_{n+1})t/\hbar) \\ &\quad - \nu_n \cos((E_n - E_{n+1})t/\hbar)) \\ \langle v_y \rangle &= 0 \end{aligned} \quad (14)$$

where $\eta_n = \gamma_n \gamma_{n+1} + \delta_n \delta_{n+1}$ and $\nu_n = \gamma_n \delta_{n+1} + \gamma_{n+1} \delta_n$, $\gamma_n = \sqrt{\frac{1}{2} + \xi_n}$ and $\delta_n = \sqrt{\frac{1}{2} - \xi_n}$ with $n = 0, 1, 2, \dots$. Several types of oscillatory motion emerge for the velocity evolution. The first term in the v_x temporal evolution is weighted by $\cos((E_n + E_{n+1})t/\hbar)$ which is the responsible of the ZB oscillatory motion. We estimate the ZB period using equation (1), which enables us to write $E_n + E_{n+1} \approx 2E_{n_0}$ [10] and then

$$T_{ZB} = \pi\hbar/|E_{n_0}| = \frac{\pi\hbar}{mc^2 \sqrt{1 + \frac{4\hbar\omega}{mc^2} n_0}}. \quad (15)$$

The second term in the v_x temporal evolution is weighted by $\cos((E_n - E_{n+1})t/\hbar)$ which lets us to extract different periodicities in the velocity temporal evolution. Using Eq. (1) again, we obtain other oscillatory scales $E_n - E_{n+1} \approx E'_{n_0}(n - n_0) + E''_{n_0}(n - n_0)^2 + \dots$, which are given by T_{CL} and T_R .

The velocity behavior is clearly illustrated in figure 1. The value $\langle v_x \rangle$ is numerically computed as a function of time for the temporal evolution of the initial wave packet $|\Psi_0\rangle$ with $n_0 = 30$ and $\sigma = 3.0$ and for an oscillator frequency $\omega = 10^3$ a. u.. (Throughout the results are generated in atomic units $m = \hbar = e = 1$). We have constructed the initial wave packet (see Eqs. 7, 8 and 9) with the levels population given in Fig. 4(a). We observe

in panel (a) that there is an oscillatory behavior for T_{ZB} -time-scale. For greater time-scales we can see in panel (b) that quasiclassical oscillations appear enveloping the ZB oscillations whose amplitude is decreasing from period to period. This behavior was previously observed in [22]. Finally in panel (c) we can clearly see a new time-scale oscillation T_R , which is enveloping the previous oscillations, and it is apparent that for $t = mT_R/2$ (for $m = 1, 2, \dots$) there is a revival of the ZB oscillation amplitude (Fig. 1 panel (c)) and the quasiclassical oscillations.

Let us illustrate this phenomenon with another example. In Fig. 2 we have considered an initial localized wave packet with a different value of the parameter $n_0 = 15$ (That is, with the levels population given in Fig. 4(b)). Then T_{CL} and T_R are smaller as ω is smaller and we observe that T_{ZB} is somewhat lower than in the above case, as we expected from equations (12), (13) and (15) respectively. Again we can observe a revival of the ZB amplitude (Fig. 2 panel (c)).

Moreover we have to stress that the appearance of revivals of ZB oscillation amplitude depends on the shape of the initial wave packet, i. e., we have to work with a localized wave-packet. If we consider a broader wave packet around lower energies $\pm E_{n_0}$, with $n_0 = 10$ and $\sigma = 20$ (Fig. 3) (see level population in Fig. 4(c)), we observe in panel (a) that there is an oscillatory behavior similar to Fig. 1 and 2 for the first quasiclassical periods where the ZB oscillation amplitude is greater when the $|\langle v_x \rangle|$ is greater. But next, we can observe a quasiclassical modulation that is disappearing in three classical periods (Fig. 3 (b)) and we will have ZB but there is no regeneration of the initial ZB amplitude (≈ 3.8 a. u.). For much longer times the revivals or quasiclassical behavior never appears (we have checked it from 0 to $10 T_R$).

Finally, in Fig. 5 we have studied the periods in terms of the parameter ω . We can see that $T_R > T_{CL} > T_{ZB}$ and T_{ZB} is almost constant for all ω . T_{CL} and T_R increase when ω decreases. In addition, when ω is smaller the temporal scales move away from each other quickly. In fact the revival of the ZB amplitude appears later. The revival of the ZB amplitude will disappear when $\omega = 0$ (which corresponds with the Lock result [2]). Note that in this limit case the ZB would be approximately 10^{-4} a. u.. These results are an extension for a bound Dirac particle of the results found for massless quasiparticles in graphene in a perpendicular magnetic field [10].

It should be noticed that the existence of revivals of the wave-packet, and consequently of the same initial quasiclassical behavior of $\langle v_x \rangle$ and ZB oscillation am-

plitude, is due to: (i) the way in which we have constructed it as a superposition of two wave-packets localized around two given eigenvalues $E_{n_0}^+$ and $-E_{n_0}^-$, and (ii) the fact that the Dirac oscillator has electron-hole symmetry ($E_n^+ = -E_n^-$) which is an essential property to obtain Eqs. (14) and (15).

Furthermore, a natural generalization of this result could be done as follows. If we consider a bound Dirac particle with a non linear spectrum E_n^\pm in n and with electron-hole symmetry, we expect that the localized Dirac particle exhibits a revival of the ZB oscillation amplitude. Although it is an open problem to prove this assertion, it could be justified since one can always consider an initial wavepacket as a superposition of two localized wavepackets, with the coefficients centered around a mean value n_0 with $|n - n_0| \ll n_0$ and obtain an analogous behavior to Eq. (14) for the temporal evolution of the velocities. We have to remark that the condition $E_n^+ = -E_n^-$ is an essential point to have definite and visible temporal scales. If the initial wave-packet is more localized and the n_0 value is higher, the revival of the ZB oscillation amplitude will be more sharp due to the fact that the regeneration will be more accurate because the Taylor expansion is more accurate too.

III. CONCLUSIONS

Summing up, we have studied the wave-packet dynamic for a Dirac oscillator demonstrating that for some particular election of the initial wave-packet there is a regeneration or revival of the ZB oscillation amplitude apart from the quasiclassical modulation of ZB in which the oscillation amplitude is decreasing. These revivals appear associated with a nonlinearity in the relativistic eigenvalue spectrum. When the frequency of oscillation is smaller the regeneration appears at longer times. In the limit of frequency zero, that is for a free Dirac particle, the regenerations disappear, due to the fact that in the case of the free Dirac particle the spectrum is continuous rather than discrete. We conjecture that this result may appear in any bound Dirac particle with electron-hole symmetry.

IV. ACKNOWLEDGMENTS

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